Introduction to Basic Calculus: Applications II

Use calculus as a tool to solve physical problems: from the conservation of mechanical energy, simple harmonic motion and elastic potential energy, to the radius of curvature, the principle of virtual work and the variable mass rope, to the kinematics of the micro-element method and the moment of inertia of rigid bodies, and the differential equations and integral ideas in series mechanics.

May 26, 2026 • Physics Olympiad • Calculus

微积分在物理学中的应用

- 力学
  - 运动 ⇒ $\vec{x}$. $\vec{v}$. $\vec{a}$. $\vec{v}=\frac{d\vec{x}}{dt}$. $\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{x}}{dt^2}$

    $\vec{x}(t)$函数
  - 静力 ⇒ 虚功原理
  - 动力 ⇒ 牛二. $\vec{F}=m\vec{a}$. $\vec{F}(t)$. $\vec{x}_0$可求

    ⇒ $\vec{a}(t)\Rightarrow\vec{v}(t)\Rightarrow\vec{x}(t)$

    动量、能量守恒

    ↳ $\Delta p=Ft$. $m\,dv=F\,dt$ $F=m\frac{dv}{dt}=\frac{dp}{dt}$
- 电磁
- 热学
- 光学
- ⋮

Example 1

$$mgh + \frac{1}{2}mv^2 = C$$
$$v = \frac{dh}{dt}$$
$$mgh + \frac{1}{2}m(\frac{dh}{dt})^2 = C$$
$$\frac{dh}{dt} = \sqrt{\frac{2(C-mgh)}{m}}$$
$$dt = \sqrt{\frac{m}{2(C-mgh)}}dh = \frac{\sqrt{2}}{2}(\frac{c}{m} - gh)^{\frac{-1}{2}}dh$$
$$\int_0^t dt = \frac{\sqrt{2}}{2}\int_{h_0}^{h_t}(\frac{c}{m} - gh)^{\frac{-1}{2}}dh$$
$$t = -\frac{\sqrt{2}}{g}(\frac{C}{m} - gh_t)^{\frac{1}{2}} + C'$$

Example 2

In simple harmonic motion:

$$\vec{F} = -k\vec{x}, \vec{v} = \frac{d\vec{x}}{dt}, \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} = \frac{-k}{m}x$$

Constructed differential equation:

$$\frac{d^2\vec{x}}{dt^2} = \frac{-k}{m}x$$

Alternative $x(t)$:

- $A\cos(\omega t + \phi)$
- $A\sin(\omega t + \phi)$
- $e^{\omega t + \phi}$

But $\frac{-k}{m} < 0$ can be excluded $e^{\omega t + \phi}$

Assume $x(t) = A\sin(\omega t + \phi)$, then:

$$v(t) = Aw\cos(\omega t + \phi)$$
$$a(t) = -Aw^2\sin(\omega t + \phi) = \frac{-k}{m}A\sin(\omega t + \phi)$$

Solution: $\omega=\sqrt{\frac{k}{m}}, \phi=\arcsin\frac{x_0}{A}$

Combined with the conservation of mechanical energy, the expression for the elastic potential energy of the spring can be derived:

When the spring oscillator is in the equilibrium position, assuming $E_{p_0}=0$, the elastic potential energy is minimum and the speed is maximum.

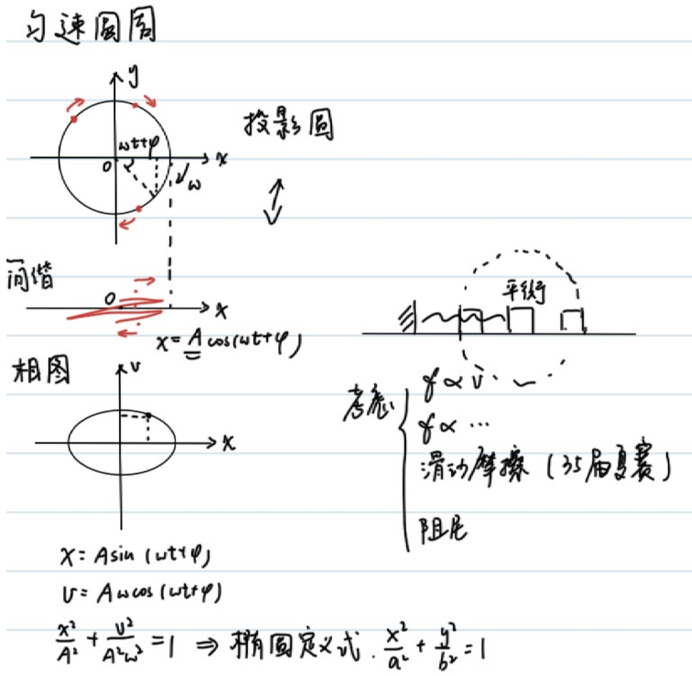
$$
\frac{1}{2}mv^2+E_p=C
$$
$$
\omega=\sqrt{\frac{k}{m}}
$$
$$
v(t)=Aw\cos(\omega t+\phi)
$$
$$
C=\frac{1}{2}m(Aw)^2
$$

Solution: $E_p=\frac{1}{2}m(Aw\sin(\omega t+\phi))^2=\frac{1}{2}kx^2$

> [!NOTE] It is well known that simple harmonic motion can correspond to uniform circular motion one-to-one
>
> A point on the circle can determine the **size and direction** of the **displacement** and **velocity** of motion.

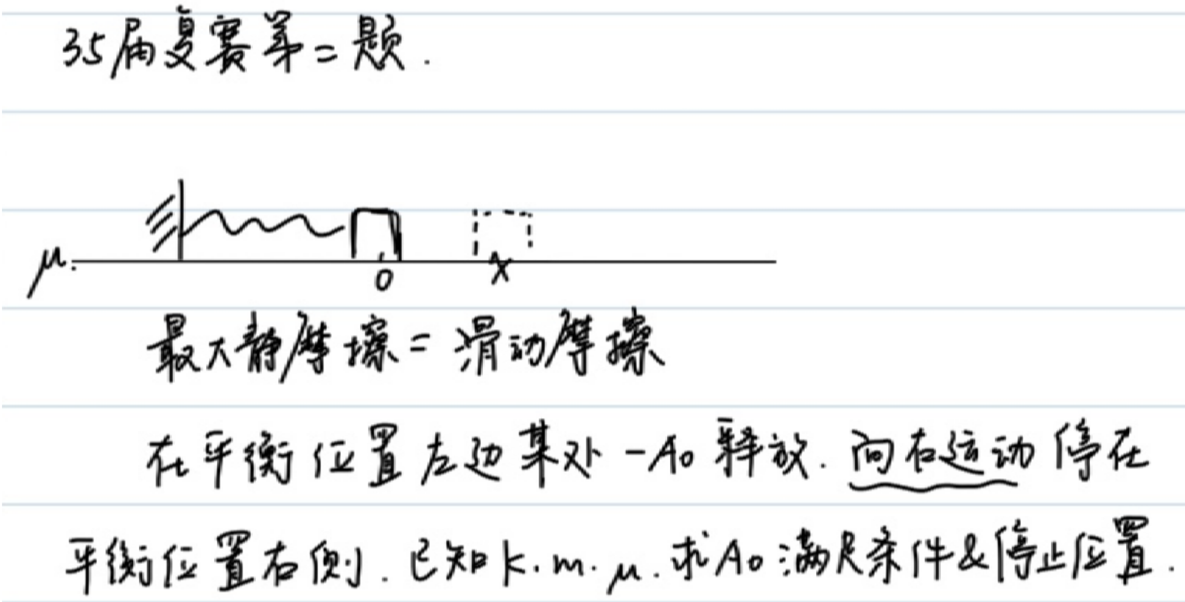
For simple harmonic motion, if you make a $v-x$ diagram, the image will be an ellipse:

$$
\begin{cases}
x=A\sin(\omega t+\phi)\\
y=A\omega\cos(\omega t+\phi)\\
\frac{x^2}{A^2}+\frac{y^2}{A^2\omega^2}=1
\end{cases}
$$



Example 3

(Second question of the 35th semi-finals - simplified version)



First, in order to be able to move in the first place, it must be true:

$$
kA_0 > \mu mg
$$
$$
A_0 > \frac{\mu mg}{k}
$$

Then list the energy conservation during motion:

$$\begin{cases} \frac{1}{2}kA_0^2 = \frac{1}{2}kx^2 + (A_0 + x)\mu mg, \\ kx < \mu mg \rightarrow x \in (0, \frac{\mu mg}{k}) \end{cases}$$

Solution:

$$A_0^2 = x^2 + \frac{2}{k}(A_0 + x)\mu mg \in (\frac{2\mu mg}{k}A_0, \frac{2\mu mg}{k}A_0 + \frac{3\mu^2 m^2 g^2}{k^2})$$

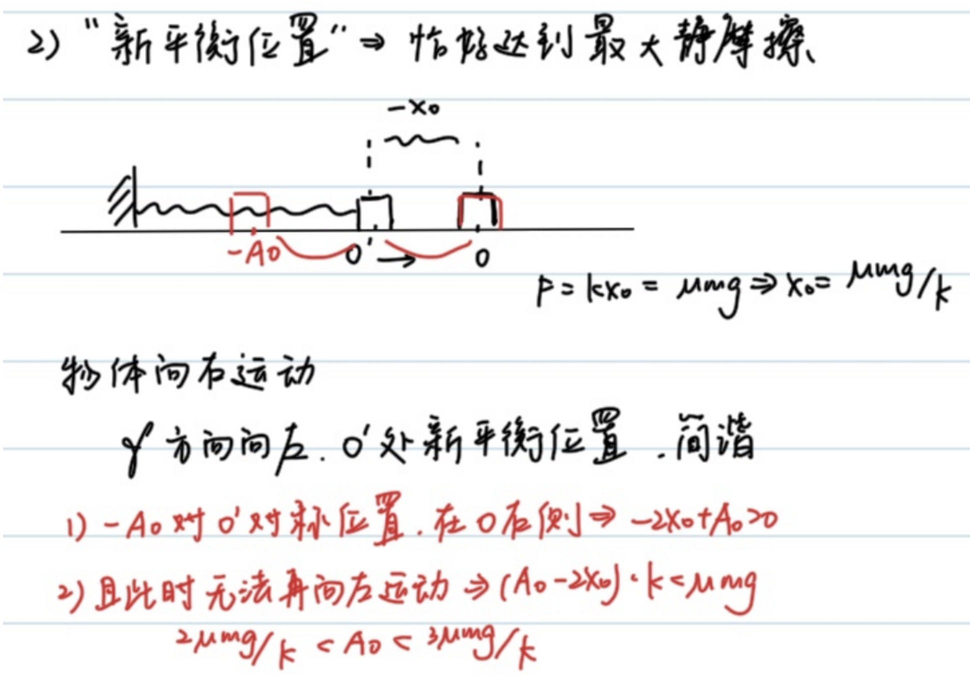
That is:

$$(A_0 - \frac{\mu mg}{k})^2 \in (\frac{\mu^2 m^2 g^2}{k^2}, \frac{4\mu^2 m^2 g^2}{k^2})$$

(Aha) Finally got:

$$A_0 \in (\frac{2\mu mg}{k}, \frac{3\mu mg}{k})$$

Another solution: (synthesize elastic force and friction force, using the symmetry of simple harmonic motion)



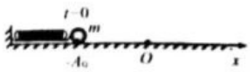
二、（40 分）如图，一劲度系数为 k 的弹簧左端固定，右端连一质量为 m 的小球；弹簧水平，它处于自然状态时小球位于坐标原点为O；小球可在水平地面上滑动，它与地面之间的动摩擦因数为μ。初始时小球速度为零，将此时弹簧相对于其原长的伸长记为$-A_o(A_o>0$，但A_o并不是已知量)。重力加速度大小为g，假设最大静摩擦力等于滑动摩擦力。

（1）如果小球至多只能向右运动，求小球最终静止的位置，和此种情形下A_o应满足的条件；

（2）如果小球完成第一次向右运动至原点右边后，至多只能向左运动，求小球最终静止的位置，和此种情形下A_o应满足的条件；

（3）如果小球只能完成n次往返运动（向右经过原点，然后向左经过原点，算 1 次往返），求小球最终静止的位置，和此种情形下A_o应满足的条件；

（4）如果小球只能完成n次往返运动，求小球从开始运动直至最终静止的过程中运动的总路程。



Original problem

Radius of curvature

A curved motion can be regarded as the sum of several circular motions, and each physical quantity satisfies:

$$\begin{cases} \vec{F} = m\frac{\vec{v}^2}{\rho}, \\ \vec{a_n} = \frac{\vec{v}^2}{\rho}\hat{n}, \\ a_n = \frac{|\vec{a}\times\vec{v}|}{|\vec{v}|}, \quad a_t = \frac{\vec{a}\cdot\vec{v}}{|\vec{v}|} \end{cases}$$

Example 4

Find the radius of curvature of parabola $y = x^2$ at $x = 0$

$$\vec{r} = (x, x^2)$$
$$\vec{v} = (\frac{dx}{dt}, 2x\frac{dx}{dt})$$
$$\vec{a} = (\frac{d^2x}{dt^2}, 2(\frac{dx}{dt})^2 + 2x\frac{d^2x}{dt^2})$$

Suppose $\frac{dx}{dt} = 1$, that is, the particle moves at a constant speed in the horizontal direction. At $x = 0$, the centripetal acceleration is along the y-axis direction and the tangential acceleration is 0:

$$a_n = |\vec{a}| = 2, \vec{v} = (1, 0), |v| = 1, \rho = \frac{v^2}{a_n} = \frac{1}{2}$$

Example 5

Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b > 0)$, find the radius of curvature at the endpoint.

If the horizontal speed is set to be constant, problems will occur at the long axis; if the vertical speed is set to be constant, problems will also occur at the short axis.

We use polar coordinate substitution:

$$\vec{x} = (a\cos(\omega t + \phi), b\sin(\omega t + \phi))$$
$$\vec{v} = (-a\omega\sin(\omega t + \phi), b\omega\cos(\omega t + \phi))$$
$$\vec{a} = (-a\omega^2\cos(\omega t + \phi), -b\omega^2\sin(\omega t + \phi))$$

At $(0, b), \omega t + \phi = \frac{\pi}{2}$.

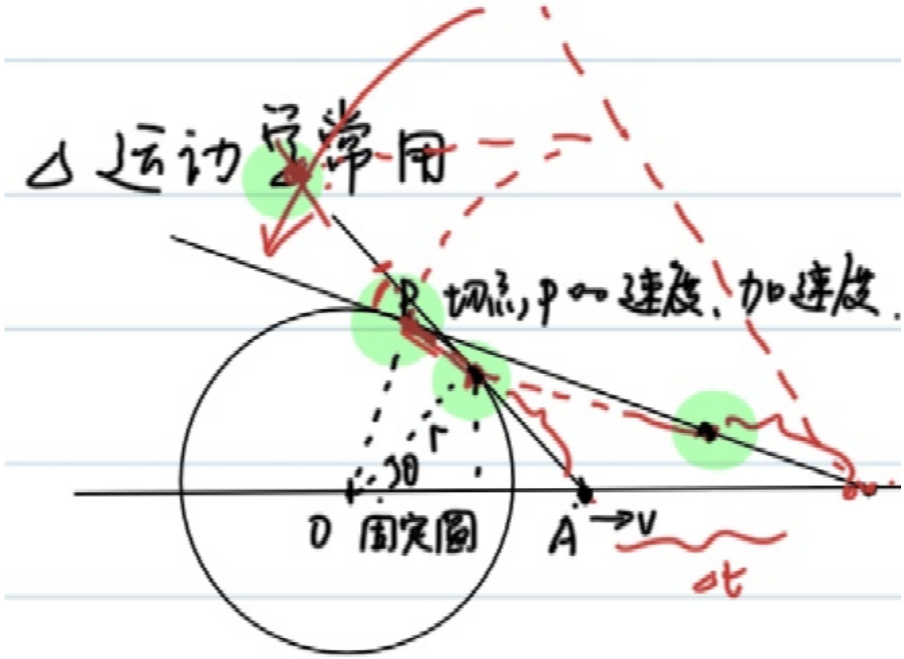
$$\vec{v} = (-a\omega, 0)$$
$$\vec{a_n} = \vec{a} = (0, -b\omega^2)$$
$$\rho = \frac{|v|^2}{|a_n|} = \frac{a^2}{b}$$

At $(a, 0), \omega t + \phi = 0$.

$$\vec{v} = (0, b\omega)$$
$$\vec{a_n} = \vec{a} = (-a\omega^2, 0)$$
$$\rho = \frac{|v|^2}{|a_n|} = \frac{b^2}{a}$$

Example 6

A long wooden stick is supported on a fixed circle. One end A moves at a constant speed on the ground with a speed of v. Find the speed of the intersection point P and the contact point E.



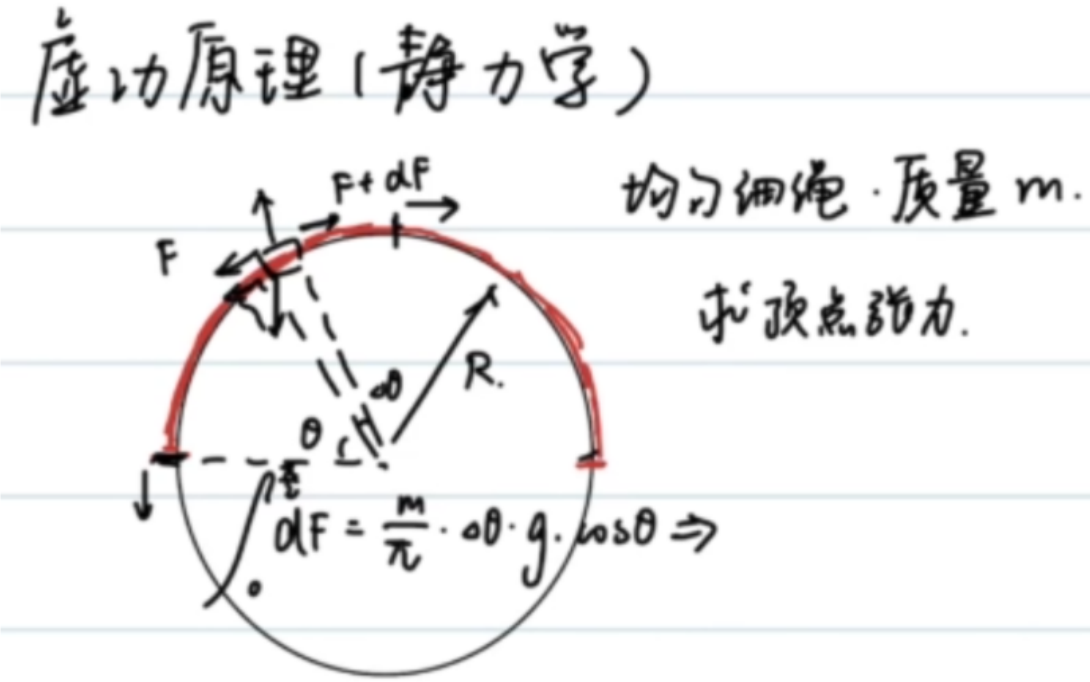
$$A(x,0)$$
$$\frac{dx}{dt}=v$$
$$P(\frac{r^2}{x},r\sqrt{1-\frac{r^2}{x^2}})$$
$$\vec{v_p}=(v\frac{-r^2}{x^2},v\frac{r^3}{x^3}\frac{1}{\sqrt{1-\frac{r^2}{x^2}}})$$

Similarly, we find the velocity of the contact point E. Note that the constraint condition of the contact point is not on the circle, but the distance to point A remains unchanged.

$$\vec{AP}=(\frac{r^2}{x}-x,r\sqrt{1-\frac{r^2}{x^2}})$$
$$|AP|=\sqrt{x^2-r^2}$$
$$\vec{l}=\frac{\vec{AP}}{|AP|}=(-\frac{\sqrt{x^2-r^2}}{x},\frac{r}{x})$$
$$E=A+|AP|$$
$$E'(t)=A'+|AP|\vec{l}'$$
$$=(v,0)+\sqrt{x^2-r^2}(-v\frac{2r^2}{x^3}\frac{1}{2\sqrt{1^2-\frac{r^2}{x^2}^2}},-v\frac{r}{x^2})$$
$$=(v(1-\frac{r^2}{x^2}),-v\frac{r}{x^2}\sqrt{x^2-r^2})$$

Example 7 (Principle of virtual work)

There is a circle with a uniform string placed on the semicircle part and the mass is m. Find the tension at the vertex.

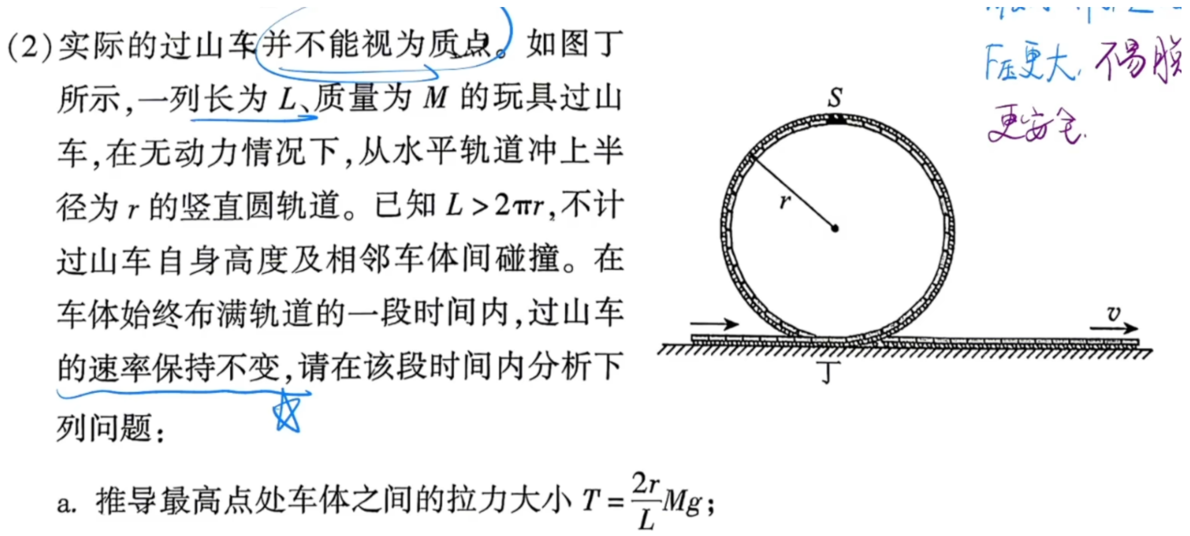


For the force analysis of countless tiny mass elements, consider the force balance along the tangential direction:

$$dF = \frac{d\theta}{\pi} mg\cos\theta$$
$$\int_0^F = \frac{mg}{\pi}\int_0^{\frac{\pi}{2}} \cos\theta d\theta$$
$$= \frac{mg}{\pi}$$

Of course, we have a highly technical approach: **Virtual Work Principle**.

Slowly pull the rope Δx, $F\Delta x = \frac{\Delta x}{\pi R} mgR$, then $F = \frac{mg}{\pi}$



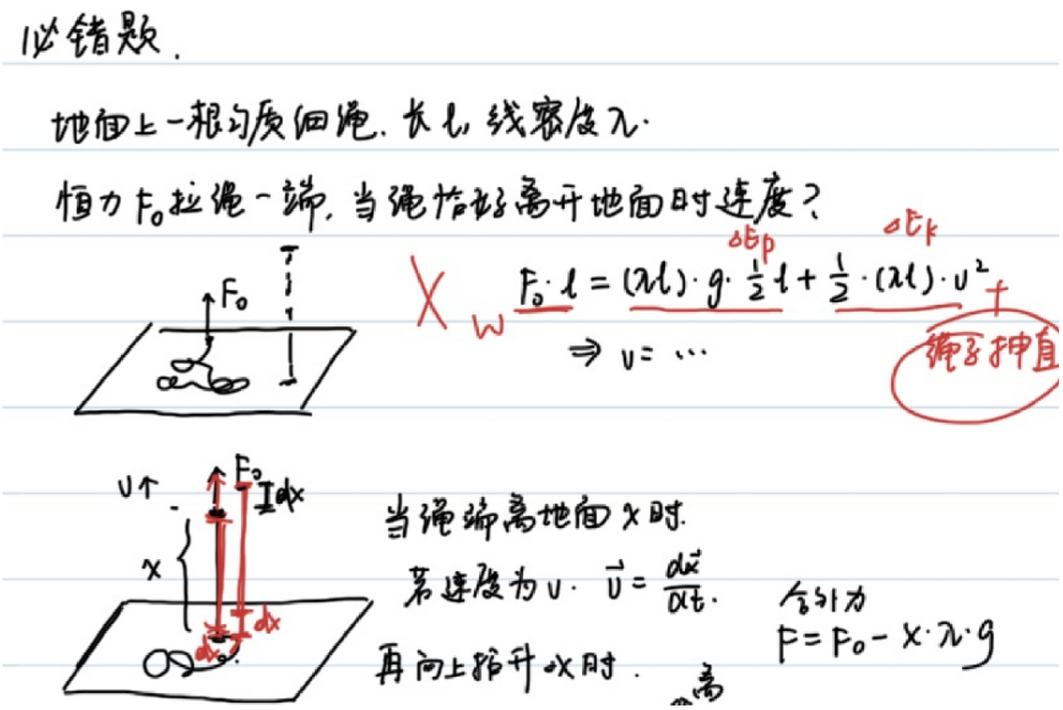
(2)实际的过山车并不能视为质点。如图丁所示，一列长为 L、质量为 M 的玩具过山车，在无动力情况下，从水平轨道冲上半径为 r 的竖直圆轨道。已知 $L>2\pi r$，不计过山车自身高度及相邻车体间碰撞。在车体始终布满轨道的一段时间内，过山车的速率保持不变，请在该段时间内分析下列问题：

a. 推导最高点处车体之间的拉力大小 $T=\frac{2r}{L}Mg$；

Using the principle of virtual work, we can easily solve the problem of 2026 Chaoyang Senior High School Second Model T19(2)

Example 8 (must be wrong)

A homogeneous thin rope on the ground, length l, linear density λ, vertical upward constant force F_0 pulls one end of the rope, the speed of the rope when it just leaves the ground v_t



$$\frac{1}{2}l\lambda gl+\frac{1}{2}l\lambda v_t^2=F_0l$$
$$v_t=\sqrt{2F_0/\lambda-gl}$$

Unfortunately, this is a **standard error**. Extra work is required to straighten a tension-free rope, and the v_t obtained by this method is too large.

When the rope end is away from the ground x, if the speed is $v, \vec{v} = \frac{d\vec{x}}{dt}$, and then lifts upward to Δx:

Momentum theorem: $Fdt = (x\lambda)dv + (dx\lambda)(v + dv)$

Approximating the second order epsilon:

$$F = \lambda x dv + \lambda v dx = \lambda \frac{d(vx)}{dt}$$

$$Fdt = \lambda d(vx)$$

But the time is unknown. If you use $dx = vdt$, you can solve this problem:

$$Fxdx = \lambda vx(xdv + vdx) = \lambda(vx)d(vx)$$

Note that F here is the resultant external force, $F = F_0 - x\lambda g$

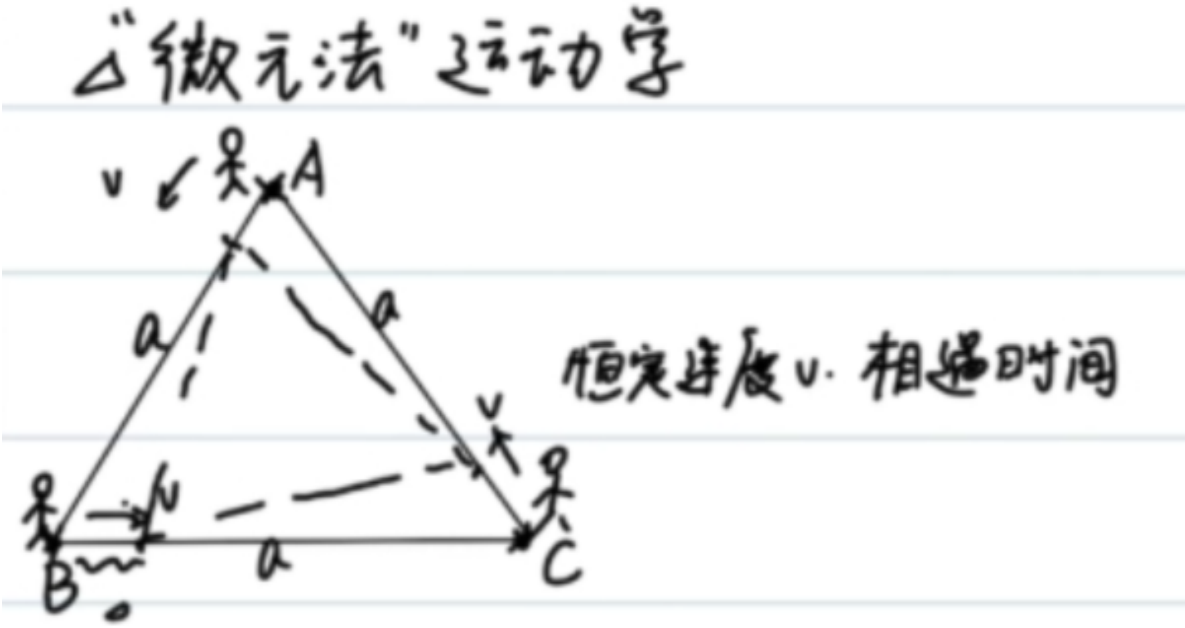
$$\int_0^l (F_0 - x\lambda g)xdx = \int_0^{lv_t} \lambda(vx)d(vx)$$

$$\frac{1}{2}F_0 l^2 - \frac{1}{3}\lambda g l^3 = \lambda \frac{1}{2}(v_t l)^2$$

$$\frac{1}{2}F_0 - \frac{1}{3}\lambda g l = \lambda \frac{1}{2}(v_t)^2$$

Solve $v_t = \sqrt{F_0/\lambda - \frac{2}{3}gl}$

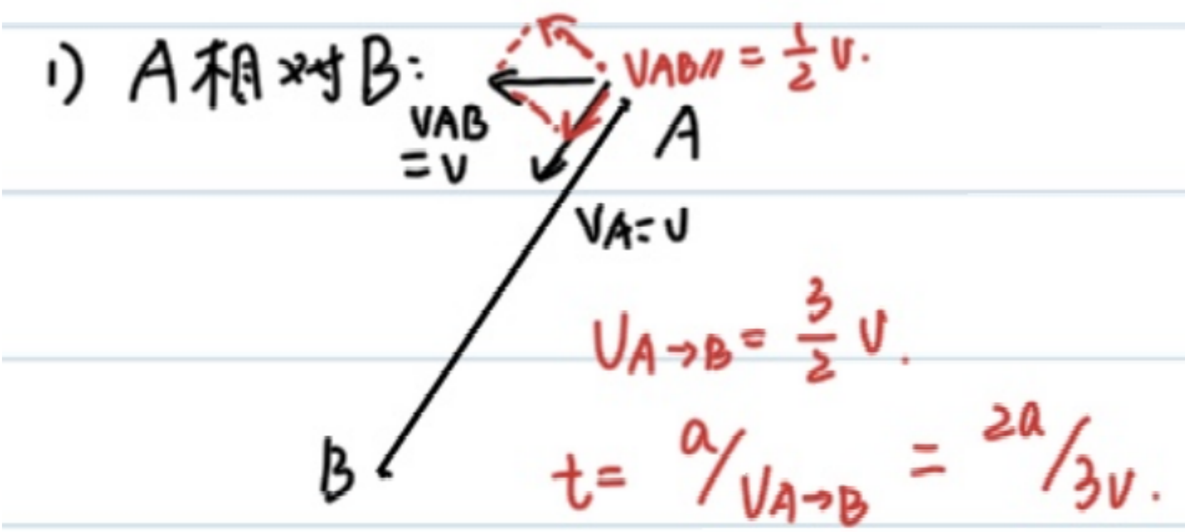
Example 9 ("Micro-element method" kinematics)



Considering symmetry, it is obvious that the figure composed of three people is always an equilateral triangle:

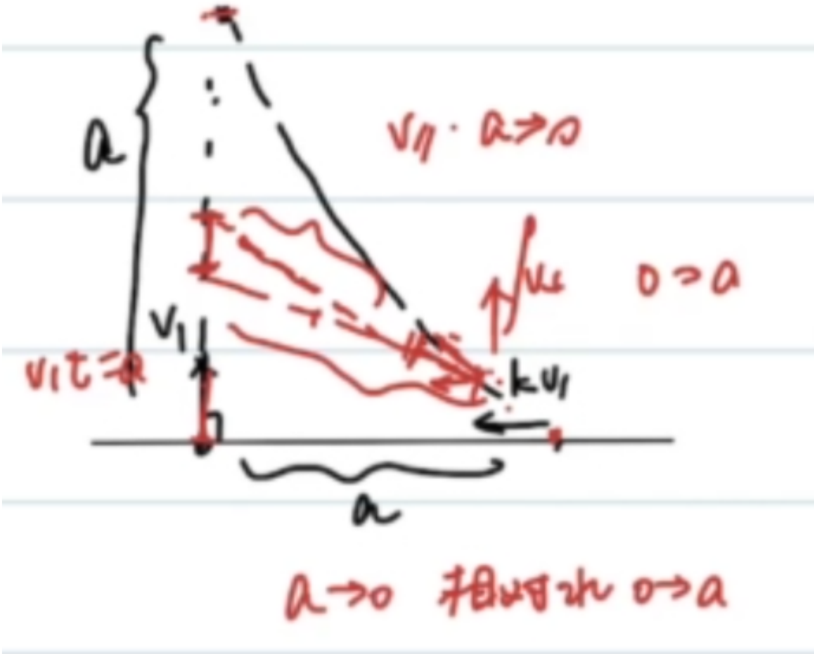
Decomposing the velocity along the connecting line direction, we can get:

$$da = -\frac{3}{2}vdt$$



$$a = \frac{3}{2}vt$$

Solve $t = \frac{2a}{3v}$



Pursuit problem

Let the angle between kv and the vertical direction be θ

$$
\begin{gathered}
vt_0 = a \\
\int_0^{t_0} kv\cos\theta dt = a \\
\int_0^{t_0} kv\sin\theta dt = a \\
dl = (-kvdt + vdt\cos\theta) \\
\int_a^0 dl = \int_0^{t_0} (-kvdt + vdt\cos\theta) \\
-a = -kvt_0 + \frac{a}{k} = -ka + \frac{a}{k} \\
k^2 - k - 1 = 0 \\
k = \frac{1+\sqrt{5}}{2}
\end{gathered}
$$

Moment of inertia

Rigid bodies such as rods, plates, etc. (the distance between particles remains unchanged) - the motion of the particle system can be decomposed into **translation of the center of mass + rotation around the center of mass**

The Sun-Earth system can be regarded as a "binary star rotating" around the center of mass, and the center of mass of the two can be considered to be in translation around the center of the Milky Way (only for multiple mass points, there is the term rotation).

The sun/earth motion can be regarded as the translational motion of the sun-earth system around the galactic center + the rotation of the sun-earth around the center of mass.

Therefore, the force will produce translational motion with respect to the center of mass; the torque of the force will cause rotation.

Newton’ s second law of point system

$$\boxed{(\sum m)a_c = \sum F}$$

Koenig (center of mass motion) theorem

$$\boxed{\sum \frac{1}{2}m_i v_i^2 = \frac{1}{2}Mv^2 + \sum \frac{1}{2}m_i v_{ic}^2}$$

Through a_c, it is not difficult to find v_c; now we hope to find the angular velocity of all particles rotating around the center of mass ω through the moment of the combined external force.

Torque $M = Fr$, angular acceleration $\beta = \frac{d\omega}{dt}$.

Definition: $M = I\beta$, where I is the moment of inertia.

$$\boxed{I = \sum m_i r_i^2}$$

The calculation formula is as above, where r_i is the distance from particle i to the center of mass.

Example 10

Calculate the moment of inertia of a uniform rod about its center

$$\begin{aligned} I &= 2\int_0^{\frac{l}{2}} \frac{dx}{l} mx^2 \\ &= \frac{1}{12}ml^2 \end{aligned}$$

Example 11

Calculate the moment of inertia of a uniform rod about one end

$$I = \int_0^l \frac{dx}{l} mx^2 = \frac{1}{3}ml^2$$

Parallel axis theorem of moment of inertia

The moment of inertia of a rigid body about any axis of rotation is equal to the sum of the moment of inertia of a rigid body about an axis parallel to the center of mass and the product of the total mass of the rigid body and the square of the vertical distance between the two axes.

$$\boxed{I = I_c + Md^2}$$

For example 11, $I = I_c + m(\frac{1}{2}l)^2 = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2$

Example 12

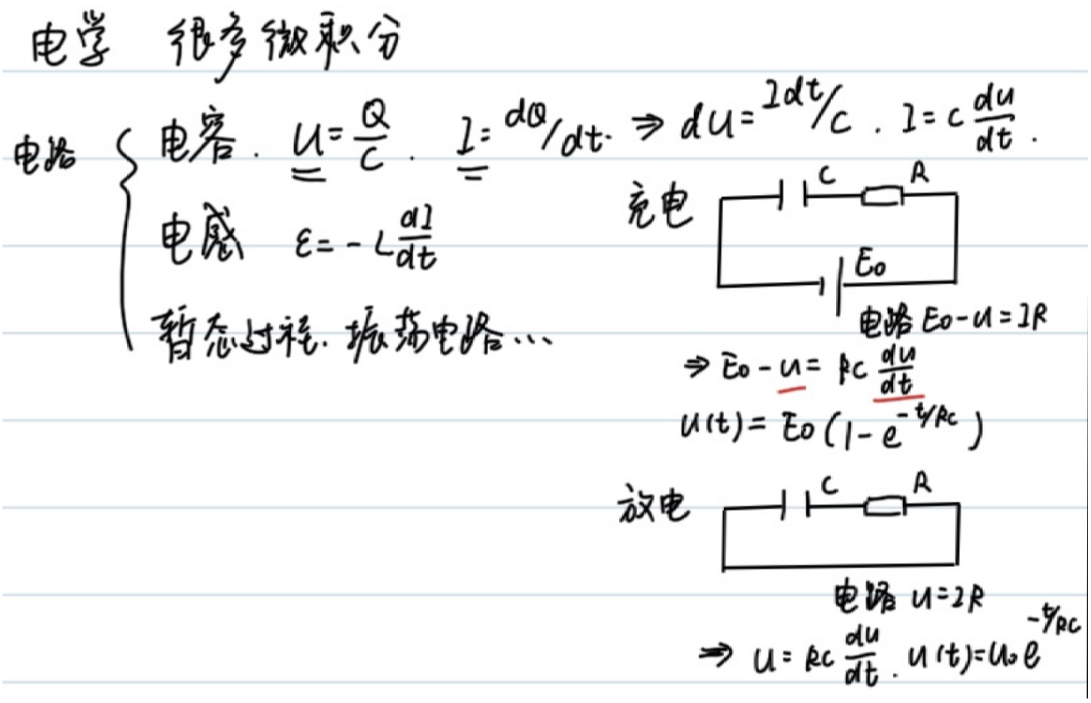
Find the moment of inertia of a uniform circular plate about an axis perpendicular to its center.

$$\begin{aligned} I &= \int_0^R \frac{2\pi r dr}{\pi R^2} mr^2 \\ &= \frac{2m}{R^2} \int_0^R r^3 \\ &= \frac{1}{2} mR^2 \end{aligned}$$

Analogy

Force	Torque
m	$I = \sum m_i r_i^2$
v	ω
a	β
$F = ma$	$M = I\beta$
$E = \frac{1}{2}mv^2$	$E = \frac{1}{2}I\omega^2$

Prospect



Summary

This article uses a series of examples as clues to show how calculus can become a unified language for solving physical problems.

- **Differential equations and conservation of mechanical energy (Example 1, Example 2)**: List the differential equation of $\frac{dh}{dt}$ or $\frac{d^2\vec{x}}{dt^2}$ based on energy conservation, and then solve it by separating variables or trial solutions. In simple harmonic motion, we deduced $\omega = \sqrt{k/m}$ from $x(t) = A\sin(\omega t + \phi)$, and derived the elastic potential energy $E_p = \frac{1}{2}kx^2$ based on the conservation of mechanical energy. At the same time, we established the correspondence between simple harmonic motion, uniform circular motion, and v-x ellipse diagrams.
- **Symmetry of simple harmonic motion (Example 3)**: In vibration problems involving friction, energy conservation is used together with the technique of "synthesizing elasticity and friction, and utilizing symmetry" to obtain the value range of the initial amplitude $A_0 \in (\frac{2\mu mg}{k}, \frac{3\mu mg}{k})$.
- **Radius of Curvature (Example 4-Example 6)**: Treat curved motion as instantaneous circular motion, use $\rho = \frac{v^2}{a_n}$ and $a_n = \frac{|\vec{a}\times\vec{v}|}{|\vec{v}|}$ to find the radius of curvature of the end points of parabola and ellipse; and avoid the trap of divergence of velocity components through appropriate parameterization (such as polar coordinate substitution).
- **The principle of virtual work and the problem of variable mass (Example 7 and 8)**: The principle of virtual work can quickly find the tension at the vertex of the semicircular rope $F = \frac{mg}{\pi}$; and for variable mass problems such as "ropes off the ground", the momentum theorem $F\,dx = \lambda\,(vx)\,d(vx)$ must be used to deal with the "extra work of straightening the

tension-free rope", otherwise it will fall into the standard error given by energy conservation, and the correct result is $v_t = \sqrt{F_0/\lambda - \frac{2}{3}gl}$.

- **Micro-element method kinematics (Example 9)**: Use symmetry and velocity decomposition along the connecting direction to deal with the catch-up problem, and turn complex trajectories into simple micro-element relationships.
- **Moment of inertia (Example 10-Example 12)**: Decompose the motion of the rigid body into "translation of the center of mass + rotation around the center of mass". The moment of inertia $I = \sum m_i r_i^2$ is derived from Newton's second law of the particle system and Koenig's theorem. The moment of inertia of the rod and circular plate is calculated and verified by the parallel axis theorem $I = I_c + Md^2$. Finally, an analogy table of the amount of translation and the amount of rotation is given.

The core idea throughout the text is: **First select appropriate physical conservation quantities or dynamic equations, and then transform them into computable mathematical problems through differential modeling, integral solution or micro-element analysis**. Calculus is not only a calculation tool, but also provides a perspective for understanding mechanical structures.