Bounded Monotone Sequence Transformations Eventually Repeat

(2024 • Xuzhou Simulation) For the sequence $P: a_1, a_2, \cdots, a_n$ where each item is a positive integer, define transformations T_1 and T_1 to transform the s

April 18, 2026 • High School Mathematics • Sequence

(2024 • Xuzhou Simulation) For the sequence $P: a_1, a_2, \cdots, a_n$ where each item is a positive integer, define transformations T_1 and T_1 to transform the sequence P into the sequence $T_1(P): n, a_1-1, a_2-1, \cdots, a_n-1$. For the sequence $Q: b_1, b_2, \cdots, b_m$ in which each item is a non-negative integer, define $S(Q)=2(b_1+2b_2+\cdots+mb_m)+b_1^2+b_2^2+\cdots+b_m^2$, define transformations T_2, T_2 Arrange the items in the sequence Q from large to small**, and then **remove all zero items** to obtain the sequence $T_2(Q)$.

(1) If the sequence P_0 is $2,4,3,7$, find the value of $S(T_1(P_0))$.

(2) For the finite sequence P_0 where each item is a positive integer, let $P_{k+1}=T_2(T_1(P_k)), k \in \mathbb{N}$.

(i) Explore the relationship between $S(T_1(P_0))$ and $S(P_0)$;

(ii) Proof: $S(P_{k+1}) \leq S(P_k)$.

(1) $T_1(P_0): 4,1,3,2,6$,
$S(T_1(P_0))=2(1\times4+2\times1+3\times3+4\times2+5\times6)+4^2+1^2+3^2+2^2+6^2=2\times53+66=172$

(2)(i)

$$\begin{gathered}S(T_1(P_0))\\ =2[n+2(a_1-1)+3(a_2-1)+\ldots+(n+1)(a_n-1)]+n^2+\sum_{i=1}^{n}(a_i-1)^2\\ =2[n-\frac{n(n+3)}{2}]+n^2+n+2(a_1+2a_2+\ldots+na_n)+2(a_1+a_2+\ldots+a_n)+\sum_{i=1}^{n}a_i^2-2(a_1+a_2+\\ =2(a_1+a_2+\ldots+a_n)+\sum_{i=1}^{n}a_i^2\\ =S(P_0)\end{gathered}$$

(ii)

From (i) we know: $S(T_1(P_0))=S(P_0)$, only T_2 causes S to become smaller:

Proof below: $S(T_2(A)) < S(A)$.

The T_2 operation** will not change the size of the sum of squares**, but putting the large number in the front will have a smaller coefficient; putting the small number in the back will have a larger coefficient, which is equivalent to a smaller **reverse sum**.

> [!TIP] Pay attention to the operation sequence of T_2 Arrange the terms in the sequence Q from large to small**, and then **remove all zero terms**

Although the final results of **Remove zeros first** and **Sort first** are the same, this order can assist us in the writing process.

典例 2　解：(1)依题意，P_0：2，4，3，7，$T_1(P_0)$：4，1，3，2，6.

卡壳点：严格按照题目中定义的运算求解.

$S(T_1(P_0))=2(4+2\times1+3\times3+4\times2+5\times6)+16+1+9+4+36=172$.

(2)(ⅰ)记 $P_0:a_1,a_2,\cdots,a_n(a_1,a_2,\cdots,a_n\in\mathbf{N}^*)$，

$T_1(P_0):n,a_1-1,a_2-1,\cdots,a_n-1$，

$S(T_1(P_0))=2[n+2(a_1-1)+3(a_2-1)+\cdots+(n+1)(a_n-1)]+n^2+(a_1-1)^2+(a_2-1)^2+\cdots+(a_n-1)^2$，

$S(P_0)=2(a_1+2a_2+3a_3+\cdots+na_n)+a_1^2+a_2^2+\cdots+a_n^2$，

卡壳点：按照所给的运算逐步求解.

$S(T_1(P_0))-S(P_0)=2n+2a_1+2a_2+\cdots+2a_n-4-6-\cdots-2(n+1)+n^2-2a_1-2a_2-\cdots-2a_n+n=n^2+3n-\frac{(2n+6)\cdot n}{2}=0$，所以 $S(T_1(P_0))=S(P_0)$.

(ⅱ)设 A 是每项均为非负整数的数列 $a_1,a_2,\cdots,a_n$，

当存在 $1\leqslant i<j\leqslant n$，使得 $a_i\leqslant a_j$ 时，交换数列 A 的第 i 项与第 j 项得到数列 B，

则 $S(B)-S(A)=2(ia_j+ja_i-ia_i-ja_j)=2(i-j)(a_j-a_i)\leqslant0$，当存在 $1\leqslant m<n$，使得 $a_{m+1}=a_{m+2}=\cdots=a_n=0$ 时，若记数列 $a_1,a_2,\cdots,a_m$ 为 C，则 $S(C)=S(A)$，

因此 $S(T_2(A))\leqslant S(A)$，从而对于任意给定的数列 P_0，

由 $P_{k+1}=T_2(T_1(P_k))(k=0,1,2,\cdots)$，$S(P_{k+1})\leqslant S(T_1(P_k))$，

由(ⅰ)知 $S(T_1(P_k))=S(P_k)$，所以 $S(P_{k+1})\leqslant S(P_k)$.

Reference answer

The answer is relatively normal, which is equivalent to proving the sorting inequality

Why is it more difficult to write **remove zeros first**? Because if **remove zeros first**, if there are 0s in the middle, then the value of S may be adapted. And **sort first** makes 0s at the end, and removing zeros does not affect S.

Finally, I’ ll add one more question, taken from the 2026 Beijing No. 2 Middle School Grade 2 (Part 2) Grade 4 exam:

(III) Prove that for any finite sequence A_0 whose terms are positive integers, there exists a positive integer K such that when $k \geq K,\ S(A_{k+1}) = S(A_k)$.

> [!TIP] Consider the boundedness, monotonicity and discreteness of S

(21) (2025 Daxing) (15 points for this question)

For a positive integer n, define $T(n)$ as the sum of the squares of each digit of n. It is known that the infinite sequence $\{a_n\}$ satisfies: a_1 is a positive integer, and for any $n \in \mathbb{N}^*$, $a_{n+1} = T(a_n)$.

(I) If $a_1 = 2$, find the value of a_2, a_3, a_4, a_5;

(II) If a_1 is a three-digit number, is there a_1 such that a_1, a_2, a_3 is incremented? If it exists, find all the values of a_1 that meet the conditions; if it does not exist, explain the reason;

(III) Prove: For any positive integer a_1, there are always positive integers K and T, such that for any $n > K, a_{n+r} = a_n$.

所以 $a_2 < a_1$，与 $a_2 > a_1$ 矛盾． ……5 分

综上，不存在 a_1，使得 a_1, a_2, a_3 递增．

（III）当 $a_n \geqslant 1000$ 时，设 a_n 是一个 k 位数，则 $a_n \geqslant 10^{k-1}$（$k \geqslant 4$），

所以 $a_{n+1} = T(a_n) \leqslant k \cdot 9^2 = 81k$.

当 $k \geqslant 4$ 时，$81k < 10^{k-1}$.

所以 $a_{n+1} = T(a_n) < a_n$.

所以存在 $N > n$，且 $a_N < 1000$. ……3 分

当 $a_n < 1000$ 时，$a_{n+1} = T(a_n) \leqslant 3 \cdot 9^2 = 243$，

所以 $a_{n+1} \in S = \{1, 2, 3, \cdots, 243\}$.

因为数列 $\{a_n\}$ 为无穷数列，

集合 $S = \{1, 2, 3, \cdots, 243\}$ 为有限集合，且 $a_{n+1} = T(a_n)$，

所以数列 $\{a_n\}$ 在集合 S 中的取值必然会出现重复．

所以存在当 $p > q$ 时，有 $a_p = a_q$.

取 $K = q$，则 $T = p - q$.

所以对任意的正整数 a_1，

总存在正整数 K 和 T，使得对任意的 $n > K$，$a_{n+T} = a_n$. ……6 分

Reference answer to part III

(III) of the two questions are exactly the same, both making use of **discreteness** and **boundedness**, leaving it to the readers to think for themselves.