A New-Definition Sequence Problem from the 2025 Haidian Grade 11 Final

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Given a positive integer $n(n \geq 3)$, if the sequence $A_n: a_1, a_2, ..., a_n$ satisfies the following two properties at the same time, then the sequence A_n is called the $P(n)$ sequence:

1. $\{a_1, a_2, .., a_n\} = \{1, 2, ..., n\}$
2. For any $i \in \{1, 2, ..., n-2\}$, there is always $j \in \{i+1, i+2, ..., n\}$ such that $|a_j - a_i| = 2$. Let the number of $P(n)$ sequences be b_n

(1) Write two $P(3)$ sequence A_3

(2) If A_n is the $P(n)$ sequence, find the value of a_1;

(3) Find the maximum value of $\frac{b_{n+1}}{b_n}$.

(1)Easy to write:

$$\{1,2,3\},\{1,3,2\},\{3,1,2\},\{3,2,1\}$$

(2) From (1), we know that not all numbers can be used as a_1, and the middle number (2) will cause problems (|x-2|=2, x is out of range).

Enumeration is simple:

n=3, $a_1 = 1 \text{ or } 3$

Make a few simple observations:

1. $a \to n+1-a$, will not affect conditions 1 and 2
2. If $1, 2, 3, ..., n-1, n$ is feasible A_n, then $n, n-1, n-2, ..., 1$ is also feasible

Digging deeper into (1), we guess that if $a_1 \pm 2 \in \{1, 2, 3, ..., n\}$, a contradiction may be derived.

When $3 \leq n \leq n-2$:

Note $B = \{a_1 - 2, a_1 - 4, a_1 - 6, ..., t\}$, where $t = \begin{cases} 1, & a_1 \text{is an odd number} \\ 2, & a_1 \text{is an even number} \end{cases}$

$C=\{a_1+2,a_1+4,a_1+6,...,s\}$, where $s=\begin{cases}n, & a_1\text{Same as n parity}\\ n-1, & a_1\text{different from n parity}\end{cases}$

According to 2, a_1 can continuously generate items in B,C, then the last two items of A_n must be numbers in B,C, with the same parity as a_1.

Let's assume that a_n,a_{n-1} is an odd number, use the extreme principle, and consider that the last item in A_n is an even number. When pushing backward, a contradiction will appear.

If a_n,a_{n-1} is an even number, consider the last odd term in the same way and derive a contradiction.

To sum up, $a_1=\begin{cases}1,3,(n=3),\\ 1,2,n-1,n,(n\geq 4)\end{cases}$

(2) Notice that $|a_j-a_i|=2$ and a_i,a_j have the same parity, so the odd and even columns in A_n "do their own thing".

Remember $x=\begin{cases}n,n\text{is an odd number}\\ n-1,n\text{is an even number}\end{cases}$,$y=\begin{cases}n-1,n\text{is an odd number}\\ n,n\text{is an even number}\end{cases}$

According to (2), a_{n-1},a_n must be one odd and one even.

Consider the order in which the odd sequence $\{1,3,5,...,x\}$ (a total of $\frac{x+1}{2}$) appears in A_n. In the same way as (2), the first appearance in A_n must be 1 or x

Next, the minimum or maximum value among the remaining numbers appears in sequence, in a total of $2^{\frac{x+1}{2}-1}$ order.

In the same way, the even sequence $\{2,4,6,...,y\}$ (a total of $\frac{y}{2}$ numbers) appears in the sequence in a total of $2^{\frac{y}{2}-1}$.

We then consider the absolute position of the odd column in A_n:

In the last two items of A_n, there is exactly one odd number, and the remaining odd numbers are in the first $n-2$ slots.

Select the absolute position of the odd-numbered column, and the absolute position of the even-numbered column is determined.

And because the internal relative positions of the odd and even columns have just been determined, according to the principle of step-by-step multiplication:

$$b_n=2C_{n-2}^{\frac{x+1}{2}-1}2^{\frac{x+1}{2}-1}2^{\frac{y}{2}-1}=C_{n-2}^{\frac{x-1}{2}}2^{\frac{x+y-1}{2}}$$

Classify n according to parity:

$$\begin{cases}b_{2k-1}=C_{2k-3}^{k-1}2^{2k-2},\\ b_{2k}=C_{2k-3}^{k-1}2^{2k-1}\end{cases}$$

Among them $k=2,3,4,\ldots$

It’ s not difficult to calculate $\frac{b_{2k}}{b_{2k-1}}=4$

$$\frac{b_{2k+1}}{b_{2k}}=4-\frac{2}{k}<4$$

Therefore, the maximum value of $\frac{b_{n+1}}{b_n}$ is 4, which is obtained if and only if $n=3,5,7,\ldots$.