# Class Review 1

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## Example 1: Analytical Algebra

It is known that the ellipse $C: \frac{x^2}{4}+\frac{y^2}{3}=1$ and the function $y=m^{x+1}+n(m\geq 2)$ intersect the ellipse at two points $A, B$, and find the maximum value of $n$.

Assume $A(x_0,y_0), B(-x_0,-y_0), x_0\in(0,2), y_0\in(0,\sqrt{3})$, there are:

$$\begin{cases} y_0=m^{x_0+1}+n, (1) \\ -y_0=m^{-x_0+1}+n, (2) \\ \frac{x_0^2}{4}+\frac{y_0^2}{3}=1, (3) \end{cases}$$

Considering that there are a total of four variables $x_0, y_0, m, n$ and three constraints, there are $4-3=1$ degrees of freedom.

Obviously if the scope of one of the variables can be determined, the scope of the other variables will be easily determined:

In order to **use symmetry** while obtaining **identical deformation**, consider (1)+(2) and (1)-(2), then the **sufficient and necessary conditions** are:

$$\begin{cases} -2n=m(m^{x_0}+m^{-x_0}), (4) \\ 2y_0=m(m^{x_0}-m^{-x_0}), (5) \\ \frac{x_0^2}{4}+\frac{y_0^2}{3}=1, (3) \end{cases}$$

Make a few simple observations:

- It is easy to see from (4) that $n<0$.
- In (4), $m^{x_0}>1$, then $m$ is larger, $-2n$ is smaller, and $n$ is larger
- In (5), the larger m is, the smaller 2y_0 is, and $m\geq 2$ can be eliminated by scaling $m$

So, we further perform **lossless scaling** (m=2):

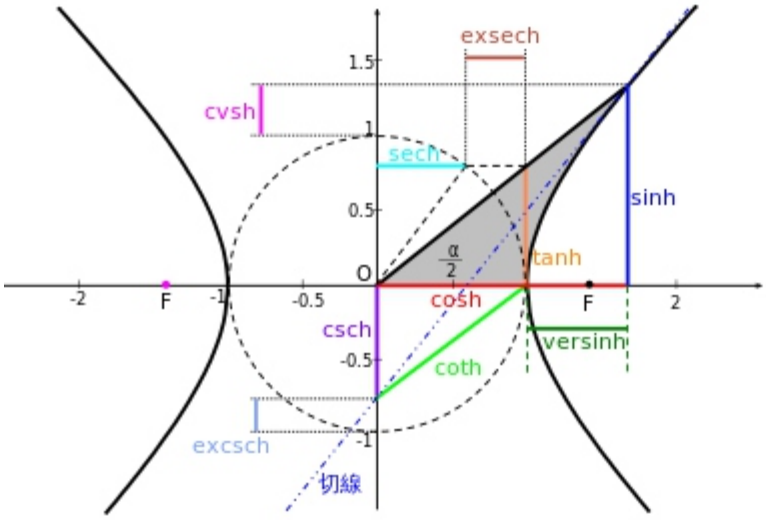
$$\begin{cases} -2n\geq 2(2^{x_0}+2^{-x_0}), (6) \\ 2y_0\geq 2(2^{x_0}-2^{-x_0}), (7) \\ \frac{x_0^2}{4}+\frac{y_0^2}{3}=1\geq\frac{x_0^2}{4}+\frac{2^{2x_0}+2^{-2x_0}-2}{3}, (\alpha) \end{cases}$$

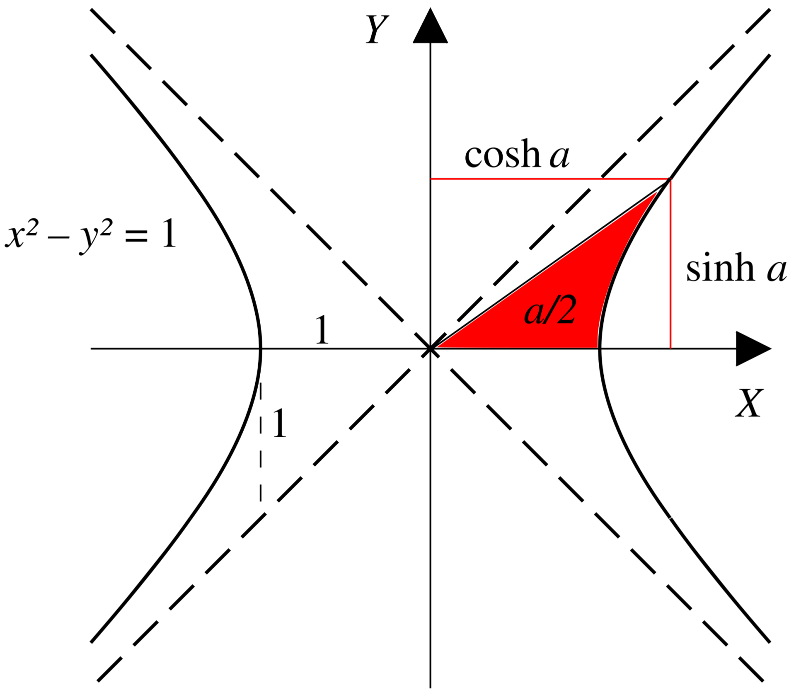
($\alpha$) is the key to solving the problem! It is not difficult to see that $\frac{x_0^2}{4} + \frac{2^{2x_0}+2^{-2x_0}-2}{3}$ is monotonically increasing with respect to $x_0$, and the solution is $x_0 \leq 1$.

So according to (6) we have:

$$n \leq -(2^{x_0} + 2^{-x_0}) \leq -\frac{5}{2}$$

## Mathematical background: hyperbolic trigonometric functions




Hyperbolic trigonometric functions are derived from hyperbola $x^2 - y^2 = 1$, so they are called **hyperbolic functions**

$$\begin{cases}\sinh x = \frac{e^x - e^{-x}}{2},\\ \cosh x = \frac{e^x + e^{-x}}{2} \geq 1,\\ \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} < 1\end{cases}$$

Similar to trigonometric functions, hyperbolic functions have corresponding identity transformation formulas:

$$\begin{aligned}\cosh^2 x - \sinh^2 x = 1,\\ \cosh^2 x + \sinh^2 x = \cosh 2x,\\ \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y\\ \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y\\ \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y\\ \cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y\\ \tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}\\ \tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}\end{aligned}$$

For details, see Wikipedia.

When dealing with Example 1 (4) (5), you can also use the analogy of $\cosh^2 x - \sinh^2 x = 1$ and consider the squares and then add them.

## Example 2

If $f(x) = a^{x-1}$ and $g(x) = \log_a x + 1$ have three intersection points, find the value range of $a$:

When $a > 1$, $f(x) = a^{x-1}$ increases monotonically, and $f(x), g(x)$ are inverse functions of each other, then if the intersection point is not on $y = x$:

Suppose the intersection point is $(x, y)$, then there must be a pair of intersection points $(y, x)$, and as long as $x \neq y$, it is inconsistent with monotonicity.

This means that $y = a^{x-1}$ as a downward convex function has two intersection points with $y = x$, which obviously leads to a contradiction:

Therefore, $0 < a < 1$, at this time, there is obviously an intersection point $(1, 1)$, and the other two intersection points cannot be on $y = x$, then the two paired intersection points are not on $y = x$, and there is a pair of $(x, y), (y, x)$ that is an intersection point that meets the meaning of the question:

$$\begin{cases}a^{x-1} = y, (1)\\ a^{y-1} = x, (2)\end{cases}$$

$$a^{x-1} + a^{y-1} = x + y,$$
$$a^{x-1} - a^{y-1} = y - x$$

Let $k = \ln a < 0, x - 1 = u, y - 1 = v$, then:

$$e^{ku} + e^{kv} = u + v + 2, (3)$$
$$e^{ku} - e^{kv} = v - u, (4)$$

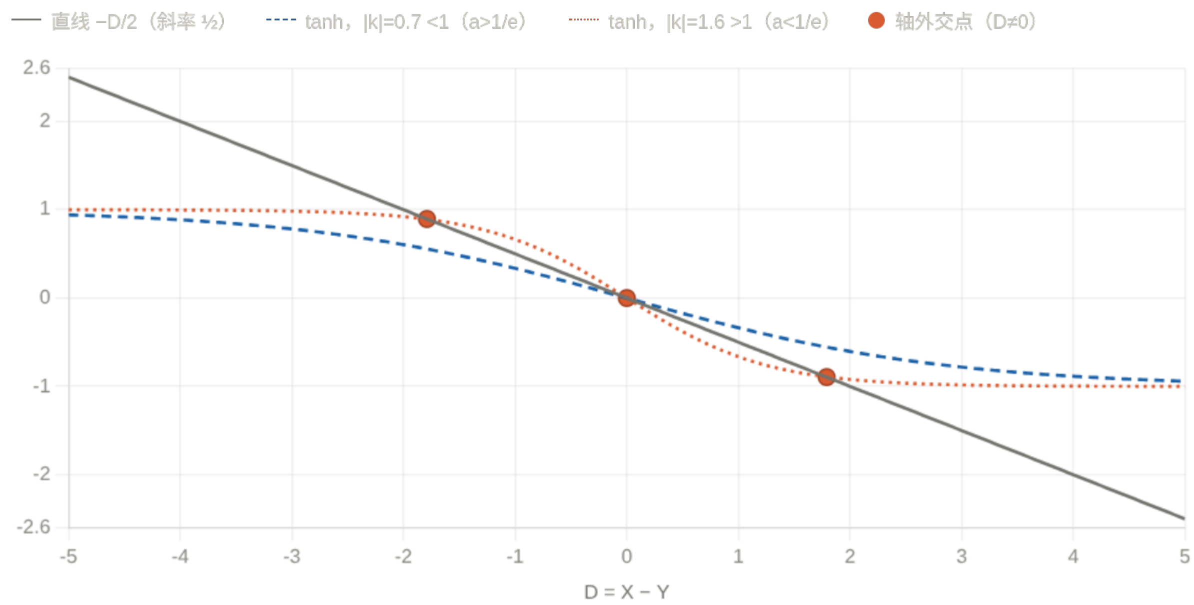
Order $u + v = S, u - v = D$ again

$$\begin{cases} e^{\frac{kS}{2}}(e^{\frac{kD}{2}} + e^{-\frac{kD}{2}}) = S + 2, (5) \\ e^{\frac{kS}{2}}(e^{\frac{kD}{2}} - e^{-\frac{kD}{2}}) = -D, (6) \end{cases}$$

(5)/(6):

$$\boxed{\tanh(\frac{kD}{2}) = -\frac{D}{S+2}}$$

The above are the necessary conditions for the existence of $x, y$.




When $S$ is fixed, $y = -\frac{1}{S+2}D$ is a straight line, considering that it has three intersection points with $y = \tanh(\frac{kD}{2})$.

Obviously, it can be seen from the image that the slope of $y = \tanh\frac{kD}{2}$ at $D = 0$ is $\frac{k}{2}$.

Then if $-\frac{1}{S+2} > \frac{k}{2}$, then the straight line and the hyperbolic tangent function will have three intersection points.

If $-\frac{1}{S+2} \leq \frac{k}{2}$, then there is only one intersection point between the straight line and the hyperbolic tangent function.

From (3):

$$S + 2 = e^{ku} + e^{kv} \geq 2e^{k\frac{S}{2}}$$

If $S < 0$, then $2 > S + 2 \geq 2e^{k\frac{S}{2}} > 2$, resulting in a contradiction.

So $S \geq 0$.

Critical condition $S \to 0, k \to -1, a \to \frac{1}{e}$

Replace "ratio slope at $S \to 0$" with a **single variable, global** argument to completely avoid the trouble of entanglement of two unknown quantities. The cleanest thing is to go back to a fixed point language.

Assume $f(x) = a^{x-1}$ $(0 < a < 1)$, the axis outer pair is the fixed point of $g(x) = f(f(x))$ except $x = 1$. inspection

$$\psi(x) = f(f(x)) - x, \qquad \psi(1) = 0.$$

**Step 1: $\psi$ Up to 3 zero points. ** To calculate the derivative, record $w(x) = f(x) + x - 2$, then

$$\psi'(x) = f'(f(x))f'(x) - 1 = (\ln a)^2 \, a^{\,w(x)} - 1.$$

Moreover, $w'(x) = a^{x-1} \ln a + 1$ is strictly increasing, so $w$ first decreases and then increases and has a unique minimum. Therefore, when the base is less than 1, $a^{w(x)}$ first increases and then decreases. Thus $\psi'$ is also unimodal: although it tends to $-1$ at both ends, it may become positive in between. Hence $\psi'$ changes sign at most twice, and $\psi$ has a decrease–increase–decrease profile with **at most three zeros**. By inverse-function symmetry, nontrivial zeros occur in pairs, so there is at most one off-axis pair.

**Step 2: The threshold is exactly $\psi'(1) = 0$. ** At the known zero point $x = 1$,

$$\psi'(1) = f'(1)^2 - 1 = (\ln a)^2 - 1.$$

- If $(\ln a)^2 \leq 1$ (i.e. $a \geq \frac{1}{e}$): then $\psi'(1) \leq 0$. Combined with the single-peak structure, it can be verified that $\psi$ no longer crosses zero on both sides of $x = 1$ (it crosses down at $1$, and both ends are also facing $-$, and the positive peak in the middle cannot reach to create a new intersection point if it exists), so there is only one zero point** and only one intersection point for $x = 1$.
- If $(\ln a)^2 > 1$ (that is, $a < \frac{1}{e}$, because $\ln a < 0$ is $\ln a < -1$): then $\psi'(1) > 0$, $\psi$ crosses the zero point at $x = 1$. However, $\psi(x) \to +\infty$ $(x \to -\infty)$ and $\psi(x) \to -\infty$ $(x \to +\infty)$ (use $0 < a < 1$ to directly test the limits at both ends), combined with the "decrease, increase and decrease" shape caused by a single peak, $x = 1$ forces a new zero point on both sides**, which is exactly a pair of off-axis solutions.

**Step Three: Continuity Closure (replacing your $S \to 0$). ** The second step is sufficient and necessary, but if you want to clarify "why this pair of solutions was born at $a = \frac{1}{e}$": $\psi'(1) = (\ln a)^2 - 1$ continues with $a$, and when $a \uparrow \frac{1}{e}$ is $\psi'(1) \downarrow 0$, the pair of zero points are continuously merged into $x = 1$ (that is, $D \to 0$). This is the correct origin of the "$S \to 0$" phenomenon - it is the ultimate behavior of the conclusion, not a criterion used as a premise.

Therefore, it is important to:

$$\text{three intersections} \iff (\ln a)^2 > 1 \text{ and } 0 < a < 1 \iff \boxed{0 < a < \frac{1}{e}}.$$

## Example 3

(2021 Tsinghua Strong Foundation) defines $x * y = \frac{x+y}{1+xy}$, then $(...(2 * 3) * 4)...) * 21 =$__.

It is not difficult to see that the background of this question is the hyperbolic tangent function.

Let $x = \frac{u-1}{u+1}, y = \frac{v-1}{v+1}, x * y = \frac{uv-1}{uv+1}$, where:

$$\begin{cases} u = -\frac{x+1}{x-1}, \\ v = -\frac{y+1}{y-1} \end{cases}$$

Note $g(x) = -\frac{x+1}{x-1}$, then:

$$(x * y) * z = (\frac{g(x)g(y) - 1}{g(x)g(y) + 1}) * z = \frac{g(x)g(y)g(z) - 1}{g(x)g(y)g(z) + 1}$$

$$(...(2 * 3) * 4)...) * 21 = \frac{g(2)g(3)...g(21) - 1}{g(2)g(3)...g(21) + 1}$$

Among them $g(2)g(3)g(4)...g(21) = \frac{-3}{1}\frac{-4}{2}\frac{-5}{3}...\frac{-22}{20} = \frac{21 \times 22}{1 \times 2} = 231$

What you want $= \frac{230}{232} = \frac{115}{116}$