2027 Strong Foundation Mathematics: AM-GM Inequalities

This article compiles 22 typical examples of mean inequality, covering denominator substitution, incremental substitution, matching, etc., local inequalities and the best estimate under product conditions. It is suitable for systematic review of the strong foundation plan and the basic stage of mathematics competitions.

July 11, 2026 • Strong Foundation Program • Mathematics Olympiad • Inequality

Example 2.1

It is known that $a > b > c$ makes the inequality

$$\frac{1}{a-b}+\frac{1}{b-c} \geq \frac{k}{a-c}$$

The maximum value of a constant real number k is ______.

The denominator is complicated, so change the denominator:

$$a-b=x, b-c=y, a-c=x+y$$
$$\frac{1}{x}+\frac{1}{y} \geq \frac{(1+1)^2}{x+y} \Longrightarrow k \leq 4$$

Example 2.2

Assume a, b, c is a positive real number, prove:

$$\frac{a+3c}{a+2b+c}+\frac{4b}{a+b+2c}-\frac{8c}{a+b+3c} \geq 12\sqrt{2}-17.$$

The denominator is complicated, so change the denominator:

$$a+2b+c=x, a+b+2c=y, a+b+3c=z$$
$$\begin{cases} a=-x+5y-3z, \\ b=x-2y+z, \\ c=0x-y+z \end{cases}$$
$$\frac{-x+2y}{x}+\frac{4x-8y+4z}{y}+\frac{8y-8z}{z}$$
$$=-17+2\frac{y}{x}+4\frac{x}{y}+4\frac{z}{y}+8\frac{y}{z}$$
$$\geq -17+2\sqrt{2\cdot 4}+2\sqrt{4\cdot 8}=-17+12\sqrt{2}$$

Example 2.3

Given $a>b>0$, then the minimum value of $a^2+\frac{1}{b(a-b)}$ is

A. 2

B. 4

C. $2\sqrt{5}$

D. 5

Consider b first:

$$a^2+\frac{1}{b(a-b)}$$
$$\geq a^2+\frac{4}{a^2}\geq 4$$

Or consider **incremental substitution**:

$$b=x, a=x+y$$
$$(x+y)^2+\frac{1}{xy}$$
$$\geq 4xy+\frac{1}{xy}\geq 4$$

Example 2.4

(2012 Tsinghua Summer Camp) It is known that a, b, c is the side length of the three sides of the triangle $\triangle ABC$, then the following judgment is correct:

A. $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$

B. $\frac{3}{2} < \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$

C. $\frac{3}{2} < \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2$

D. $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2$

$$a < b + c \Longrightarrow \frac{2a}{a+b+c} > \frac{a}{b+c}$$

Rotate $a \to b \to c \to a$ and get:

$$b < c + a \quad \Longrightarrow \quad \frac{2b}{a+b+c} > \frac{b}{c+a}$$

$$c < a + b \quad \Longrightarrow \quad \frac{2c}{a+b+c} > \frac{c}{a+b}$$

Accumulation: $S < 2$

Let's consider the lower bound, repeat the same trick, change the denominator, and use the AM-GM inequality:

$$\begin{gathered} b+c=x, c+a=y, a+b=z \\ S=\frac{\frac{y+z-x}{2}}{x}+\frac{\frac{z+x-y}{2}}{y}+\frac{\frac{x+y-z}{2}}{z} \\ =\frac{1}{2}(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z})-\frac{3}{2} \\ \geq 3-\frac{3}{2}=\frac{3}{2} \end{gathered}$$

Choose A.

Example 2.5

If the positive number x, y satisfies $x^2 + 2xy - 1 = 0$, then the minimum value of $2x + y$ is

A. $\frac{\sqrt{2}}{2}$

B. $\sqrt{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\sqrt{3}$

$$
\begin{gathered}
2x+y=u, y=u-2x \\
x^2+2(u-2x)x-1=0 \\
3x^2-2ux+1=0 \\
\Delta=4u^2-12\geq 0 \\
u\geq\sqrt{3}(x=y=\frac{\sqrt{3}}{3})
\end{gathered}
$$

Or consider exchanging yuan:

$$
\begin{gathered}
x(x+2y)=1 \\
2x+y=\frac{3}{2}x+\frac{1}{2}(x+2y)\geq\sqrt{3x(x+2y)}=\sqrt{3}
\end{gathered}
$$

Example 2.6

If the positive number a,b,c satisfies $a(a+b+c)+bc=4$, then the minimum value of the formula $3a+2b+c$ is ______.

$$
\begin{gathered}
a^2+ab+ac+bc=(a+b)(a+c)=4 \\
3a+2b+c=2(a+b)+(a+c) \\
\geq 2\sqrt{2(a+b)(a+c)}=4\sqrt{3}
\end{gathered}
$$

Example 2.7

Assume $x,y,z>0$, then

$$
\frac{xy+2yz}{x^2+y^2+z^2}
$$

The maximum value is ______.

$$\begin{aligned}
&\frac{xy+2yz}{x^2+y^2+z^2} \\
&= \frac{xy+2yz}{x^2+\frac{1}{5}y^2+\frac{4}{5}y^2+z^2} \\
&\leq \frac{xy+2yz}{2\sqrt{\frac{1}{5}}xy+2\sqrt{\frac{4}{5}}yz} \\
&\leq \frac{\sqrt{5}}{2}
\end{aligned}$$

Example 2.8

Given $a \geq 0, b \geq 0, c \geq 0, a+b+c=1$, find the maximum value of $\sqrt{4a+1}+\sqrt{4b+1}+\sqrt{4c+1}$.

Round up the mean by averaging:

$$\sqrt{(4a+1)\frac{7}{3}} \leq \frac{4a+1+\frac{7}{3}}{2}$$

Simplification factor:

$$\begin{aligned}
\sqrt{4a+1} &\leq \frac{2\sqrt{21}}{7}a+\frac{5\sqrt{21}}{7} \\
\sqrt{4b+1} &\leq \frac{2\sqrt{21}}{7}b+\frac{5\sqrt{21}}{7} \\
\sqrt{4c+1} &\leq \frac{2\sqrt{21}}{7}c+\frac{5\sqrt{21}}{7} \\
S &\leq \frac{2\sqrt{21}}{7}(a+b+c)+\frac{5\sqrt{21}}{7}=\sqrt{21}
\end{aligned}$$

Example 2.9

(Zhejiang University) Suppose the sum of positive numbers $x_1, x_2, \ldots, x_n$ is equal to 1 ($n \geq 2$), prove:

$$\frac{1}{x_1-x_1^3}+\frac{1}{x_2-x_2^3}+\cdots+\frac{1}{x_n-x_n^3}>4$$

Consider local inequalities:

$$\frac{1}{x-x^3} \geq 4x, x \in (0,1)$$
$$\Longleftrightarrow (2x^2-1)^2 \geq 0(x=\frac{\sqrt{2}}{2})$$

Obviously, it is impossible for every x to be $\frac{\sqrt{2}}{2}$, Q.E.D.

Example 2.10

Given $a_i > 0$, and $a_1a_2a_3\cdots a_n = 1$, find the minimum value of $(a_1+2)(a_2+2)\cdots(a_n+2)$.

Still holding the hand holding the equal sign:

$$(a_1+1+1)(a_2+1+1)\cdots(a_n+1+1)$$
$$\geq 3^n\sqrt[3]{a_1a_2a_3\cdots a_n} = 3^n$$

Example 2.11

Given $x > 0$, find the minimum value of $x^2+\frac{16}{x}$.

$$x^2+\frac{8}{x}+\frac{8}{x}$$
$$\geq 3\sqrt[3]{8^2} = 12(x=2)$$

Example 2.12

Given $x > 0$, find the minimum value of $x^2+\frac{1}{x^3}$.

$$\frac{x^2}{3}+\frac{x^2}{3}+\frac{x^2}{3}+\frac{1}{2x^3}+\frac{1}{2x^3}$$
$$\geq 5\sqrt[5]{\frac{1}{3^3\cdot 2^2}} = \frac{5}{6}\sqrt[5]{72}$$

Example 2.13

Given $0 < x < 1$, find the maximum value of $x^2\sqrt{1-x}$.

The lower bound is obviously 0, there is no minimum value, consider the maximum value:

$$
\begin{gathered}
x^2\sqrt{1-x} \\
=\sqrt{x^4(1-x)} \\
=\frac{1}{2}\sqrt{x\cdot x\cdot x\cdot x(4-4x)}\le\frac{1}{2}\sqrt{(\frac{4}{5})^5}=\frac{16\sqrt{5}}{125}(x=\frac{4}{5})
\end{gathered}
$$

Example 2.14

Assume $x>-\frac{1}{2}$, then the minimum value of $f(x)=x^2+x+\frac{4}{2x+1}$ is ______.

Change denominator: $t=2x+1>0$

$$
\begin{gathered}
f(x)=(\frac{t-1}{2})^2+\frac{t-1}{2}+\frac{4}{t} \\
=\frac{t^2}{4}+\frac{4}{t}-\frac{1}{4} \\
=\frac{t^2}{4}+\frac{2}{t}+\frac{2}{t}-\frac{1}{4}\ge 3-\frac{1}{4}=\frac{11}{4}(t=2,x=\frac{1}{2})
\end{gathered}
$$

Example 2.15

Assume $a+3b=3$, and find the minimum value of the algebraic expression 3^a+9^b.

$$
\begin{gathered}
3^a+9^b=3^a+3^{2b} \\
=\frac{3^a}{2}+\frac{3^a}{2}+\frac{3^{2b}}{3}+\frac{3^{2b}}{3}+\frac{3^{2b}}{3} \\
\ge 5\sqrt[5]{\frac{3^{2a+6b}}{2^2\cdot 3^3}}=5\sqrt[5]{\frac{3^3}{2^2}}=\frac{5}{2}(15)^{\frac{3}{5}}
\end{gathered}
$$

Example 2.16

Assume $a>b>0$, and prove $\sqrt{2a^3}+\frac{3}{ab-b^2}\ge 10$.

① $\ge\sqrt{2}\cdot(2\sqrt{xy})^3+\frac{3}{xy}$

$$
=8\sqrt{2}(xy)^{\frac{3}{2}}+\frac{3}{xy}
$$

$$= 4\sqrt{2}(xy)^{\frac{3}{2}} + 4\sqrt{2}(xy)^{\frac{3}{2}} + \frac{3}{xy}$$

$$\geq 5\sqrt[5]{32} = 10$$

② $\geq \sqrt{2}(x+y)^3 + \frac{12}{(x+y)^2}$

$$= \frac{\sqrt{2}}{2}(x+y)^3 + \frac{\sqrt{2}}{2}(x+y)^3 + \frac{4}{(x+y)^2} + \frac{4}{(x+y)^2} + \frac{4}{(x+y)^2}$$

$$\geq 5\sqrt[5]{32} = 10$$

Example 2.17

Assume a, b, c is a positive real number, and $a+b+c=1$, prove: $\frac{1}{a+bc} + \frac{1}{b+ac} + \frac{1}{c+ab} \geq \frac{27}{4}$.

The equal sign is obviously $a = b = c = \frac{1}{3}$.

$$\frac{1}{a+bc}$$
$$= \frac{1}{a(a+b+c)+bc}$$
$$= \frac{1}{(a+b)(a+c)}$$
$$S \geq 3\sqrt[3]{\frac{1^3}{(a+b)(a+c)(b+c)(b+a)(c+a)(c+b)}}$$
$$\geq 3\sqrt[3]{\frac{1}{(\frac{4}{6})^6}} = \frac{27}{4}$$

Example 2.18

It is known that the positive real number a, b satisfies $ab(a+b) = 4$, then the minimum value of $2a+b$ is _____.

$$4uv = (ua)(vb)(a+b) \le (\frac{(u+1)a+(v+1)b}{3})^3$$
$$\frac{u+1}{v+1} = 2, ua = vb = a+b = k$$
$$u = 2v+1, a = \frac{k}{2v+1}, b = \frac{k}{v}$$
$$\frac{1}{v} + \frac{1}{2v+1} = 1$$
$$\implies v = \frac{1+\sqrt{3}}{2}, u = 2+\sqrt{3}$$
$$\frac{\frac{3+\sqrt{3}}{2}(2a+b)}{3} \ge \sqrt[3]{2(\sqrt{3}+1)(\sqrt{3}+2)} = 1+\sqrt{3}$$
$$2a+b \ge 2\sqrt{3}$$

Or a more attention-demanding solution:

$$(2a+b)^2 = 4a(a+b)+b^2 \ge \frac{16}{b} + b^2 \ge 12$$
$$\implies 2a+b \ge 2\sqrt{3}$$

Example 2.19

Assume $S_n = \sum_{k=1}^{n} \frac{1}{k}$, prove: $n(n+1)^{1/n} - n < S_n < n - (n-1)n^{-1/(n-1)} \quad (n > 1)$.

Key:1 pairing

$$S_n + n = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} \ge n(n+1)^{\frac{1}{n}}$$
$$n - S_n = 0 + \frac{1}{2} + \frac{2}{3} + \cdots + \frac{n-1}{n} \ge (n-1)(\frac{1}{n})^{\frac{1}{n-1}}$$

Example 2.20

Proof: $1 - \frac{1}{2012}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2013}\right) > \frac{1}{\sqrt[2012]{2013}}$.

Still note the pairing of 1:

$$\iff 2012 - (\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2013}) > 2012 \sqrt[2013]{\frac{1}{2023}}$$
$$\iff \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{2012}{2013} > 2012 \sqrt[2013]{\frac{1}{2023}}$$

Example 2.21

It is known that m, n is a positive integer, and $1 < m < n$, prove: $(1+m)^n > (1+n)^m$.

$$(1+n)^m \cdot \underbrace{1 \times 1 \times \cdots \times 1}_{n-m \text{ terms}}$$
$$< (\frac{(n-m)+m(1+n)}{n})^n = (1+m)^n$$

Example 2.22

Suppose $x_1, x_2, \ldots, x_{n+1} > 0$ satisfies $\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_{n+1}} = 1$, and verify: $x_1 x_2 \cdots x_{n+1} \geq n^{n+1}$.

Equality condition: all n.

Use the substitution method to use conditions:

$$\frac{1}{1+x_i} = a_i$$
$$x_i = \frac{1}{a_i} - 1$$
$$x_i = \frac{a_1 + a_2 + \cdots + a_{n+1}}{a_i} - 1$$
$$x_i = \frac{a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{n+1}}{a_i}$$
$$\geq \frac{n\sqrt[n]{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{n+1}}}{a_i} (i = 1, 2, 3, \cdots, n+1)$$

The cumulative multiplication of all is proved